A HILBERT SPACE APPROACH OF POINCARÉ RECURRENCE THEOREM

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Dedicated to Prof. Romulus Cristescu on his 70th birthday

ABSTRACT. The aim of this paper is to prove the following noncommutative analogue of Poincaré recurrence theorem: *Let $\mathfrak{A}$ be a unital $C^\star$-algebra and let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a $C^\star$-homomorphism of unital $C^\star$-algebras, which is invariant with respect to a state φ of $\mathcal{A}$. Then*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \varphi(\Phi^k(a^\star)a) \right| > 0$$

for every $a \in \mathfrak{A}$ with $\varphi(a) \neq 0$. Comments on further extensions are also included.

Recurrence was introduced by Poincaré [11] in connection with its study on celestial mechanics and refers to the property of an orbit to come arbitrarily close to positions already occupied. Poincaré noticed that almost all orbits of any classical Hamiltonian system in which the phase space is Euclidian are recurrent. His result, known as the *Poincaré recurrence theorem*, can be stated in a measure-theoretical form as follows: *If T is a measure-preserving transformation of a probability space (Ω, Σ, μ) and $A \in \Sigma$ then almost every $x \in A$ returns infinitely often in A.*

For details, see the monographs of H. Furstenberg [5] and U. Krengel [8]. An equivalent formulation of Poincaré recurrence theorem is the nonexistence of wandering subsets of positive measure:

(NW) $\qquad \mu(A) > 0$ implies $\mu(T^{-n}A \cap A) > 0$ for some $n \geq 1$.

See [8], p. 16. Khintchine recurrence theorem [7], [10] provides a quantitative version of (NW) : *If T is a measure-preserving transformation of a probability space (Ω, Σ, μ), then for every $A \in \Sigma$ and every $\varepsilon > 0$ there exist a relatively dense subset $\mathcal{N}$ of N such that*

$$\mu \left(T^{-n}A \cap A \right) \geq \mu \left(A \right)^2 - \varepsilon$$

for all $n \in \mathcal{N}$. Recall that a subset $\mathcal{N}$ of $\mathbb{N}$ is called *relatively dense* provided that there exists an $L > 0$ such that in every interval of natural numbers having length $\geq L$ one can find a number $n \in \mathcal{N}$.

If T is a mixing mapping then $\mu\left(T^{-n}A \cap A\right) \to \mu\left(A\right)^2$, so that the above estimate is the best possible.

Notice that while Poincaré recurrence theorem refers to single orbits, Khintchine recurrence theorem refers to configurations and yields a quite optimistic conclusion about the possibility to recover the initial state of a configuration from a sequence of observations. In fact, it asserts the nonexistence of wandering configurations.

In the framework of $C^\star-$algebra theory, the measure preserving transformations T acting on (Ω, Σ, μ) can be seen as unital $^\star-$homomorphisms

$$\Phi : L^\infty(\mu) \to L^\infty(\mu), \quad \Phi(f) = f \circ T$$

which leaves invariant the faithful normal trace

$$\varphi(f) = \int_\Omega f \, d\mu.$$

Using Hilbert space techniques, we shall prove the following noncommutative generalization of the Khintchine recurrence theorem:

Theorem A. *Let $\mathfrak{A}$ be a $C^\star-$algebra with unit $\mathbf{1}$, and let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a $C^\star-$morphism of unital algebras.*

We assume the existence of a state $\varphi \in \mathfrak{A}^\star$ such that $\varphi \circ \Phi = \varphi$.

Then for every element $a \in \mathfrak{A}$ and every $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that

$$Re\varphi(\Phi^n(a^\star)a) \geq |\varphi(a)|^2 - \varepsilon$$

for every $n \in \mathcal{N}$.

Theorem A can be further extended for arbitrary $C^\star-$algebras $\mathfrak{A}$ and Schwarz mappings $\Phi : \mathfrak{A} \to \mathfrak{A}$ i.e., linear mappings such that

$$\Phi(a)^\star\Phi(a) \leq \Phi(a^\star a) \quad \text{for every } a \in \mathfrak{A}.$$

See [9]. However, the proof of Theorem A in that case appears to be considerably more involving.

When $\mathfrak{A}$ is a $C^\star-$algebra with unit $\mathbf{1}$, and φ is a state with $\varphi\left(\mathbf{1}\right) = 1$, then the pair $(\mathfrak{A}, \varphi)$ can be thought of as a noncommutative probability space in the sense of Voiculescu [3]; in that case the elements of $Re\mathfrak{A}$ should be interpreted as real random variables.

The main result of our paper is Theorem B below, which constitutes a consequence of Theorem A to $C^\star-$dynamical systems.

By a $C^\star-$*dynamical system* we mean any triplet $(\mathfrak{A}, \varphi, \Phi)$, where $(\mathfrak{A}, \varphi)$ is a noncommutative probability space and $\Phi : \mathfrak{A} \to \mathfrak{A}$ is a homomorphism of unital $C^\star-$algebras such that $\varphi \circ \Phi = \varphi$. Any measurable dynamical system (i.e., any quadruplet (X, Σ, μ, T) as in the statement of Khintchine recurrence theorem) can be thought of as a $C^\star-$dynamical system $(L^\infty\left(\mu\right), \varphi, \Phi)$ by letting

$$\Phi\left(f\right) = f \circ T \text{ and } \varphi\left(f\right) = \int_X f \, d\mu.$$

Even in this particular case Theorem A is stronger than Khintchine's result because it refers to all random variables (not only to projections).

Theorem A yields the nonexistence of wandering random variables in the general context of $C^\star-$dynamical systems and that could explain the phenomenon of superstability in Physics.

Theorem B. (The noncommutative analogue of Poincaré recurrence theorem). *Let* $(\mathfrak{A}, \varphi, \Phi)$ *be a* $C^\star-$*dynamical system. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\varphi(\Phi^k(a^\star)a)\right| > 0$$

for every $a \in \mathfrak{A}$ *with* $\varphi(a) \neq 0$.

The idea how to derive Theorem B from Theorem A is quite easy and goes back to Koopman and von Neumann. See [5] or [8]. In fact, they noticed that given a bounded sequence $(a_n)_n$ of positive numbers, the following two conditions are equivalent:

$i)$ $\liminf\limits_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k > 0$;

$ii)$ $\liminf\limits_{n\to\infty} \frac{\text{Card}\,\{k\in[0,n]:\, a_k>\varepsilon\}}{n} > 0$ for some $\varepsilon > 0$.

1. Proof of Theorem A

The proof of Theorem A is based on the following more general result:

1.1. Lemma. Let H be a vector space endowed with a hermitian form $\langle, ., \rangle$ and let $||.||$ be the corresponding seminorm. We assume the existence of a linear operator $U : H \to H$ such that

$$\begin{aligned} ||Ux|| &= ||x|| \quad \text{for every } x \in H \\ Uv &= v \quad \text{for some } v \in H, \ ||v|| = 1. \end{aligned}$$

Then for every $x \in H$ and every $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{N}$ of $\mathbb{N}$ such that

$$Re < U^n x, x > \geq |< x, v >|^2 - \varepsilon$$

for every $n \in \mathcal{N}$.

Proof. By the mean ergodic theorem there exists a projection P on $\{z : Uz = z\}$ such that for every $\varepsilon > 0$ and every $x \in H$ we can find a rank N for which

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} U^k x - Px \right\|^2 \leq \varepsilon/2.$$

Put

$$x_N = \frac{1}{N} \sum_{k=0}^{N-1} U^k x.$$

Because U is a contraction and $UP = P$, we have also

$$\left\|U^l x_N - Px\right\|^2 \leq \varepsilon/2$$

for every $l \in \mathbb{N}$. Then $\left|\left|U^l x_N - x_N\right|\right|^2 \le 2\varepsilon$ for every $l \in \mathbb{N}$, which yields

$$Re \left\langle U^l x_N, x_N \right\rangle \ge ||x_N||^2 - \varepsilon$$

for every $l \ge N - 1$. On the other hand,

$$0 \le ||x_N - <x_N, v> v||^2 = ||x_N||^2 - |<x_N, v>|^2$$

and

$$<x_N, v> = <x, v>$$

because $U^\star v = v$; see [12], § 144. Then

$$|<x, v>|^2 \le ||x_N||^2 \le \varepsilon + Re \left\langle U^l x_N, x_N \right\rangle.$$

Or,

$$\begin{aligned} Re \left\langle U^l x_N, x_N \right\rangle &= \frac{1}{N^2} \sum_{j,\,k=0}^{N-1} Re \left\langle U^{k+l} x, U^j x \right\rangle = \\ &= \frac{1}{N^2} \sum_{j,\,k=0}^{N-1} Re \left\langle U^{l+k-j} x, x \right\rangle \end{aligned}$$

so that for each integer $n \in \mathbb{N}^\star$ there must exist integers $j(n), k(n) \in [0, N-1]$ for which

$$Re \left\langle U^{nN+k(n)-j(n)} x, x \right\rangle \ge |<x, v>|^2 - \varepsilon.$$

Moreover, the set $\mathcal{N} = \{nN + k(n) - j(n) : n \in \mathbb{N}^\star\}$ is relatively dense in $\mathbb{N}$ because

$$(n-1) N \le nN + k(n) - j(n) \le (n+1) N$$

i.e., $\mathcal{N}$ contains an element in every interval of length $2N$. ∎

To derive Theorem A from Lemma 1.1 it suffices to think at $\mathfrak{A}$ endowed with the GNS-hermitian product

$$<x, y> = \varphi(y^* x)$$

and to choose as U the mapping Φ and as v the unit of $\mathfrak{A}$.

2. The Jordan algebra variant

Let $\mathfrak{A}$ be a $C^\star$-algebra. A *Jordan algebra* in $\mathfrak{A}$ is any vector subspace J of $Re\,\mathfrak{A}$ such that for every $x, y \in J$ the *Jordan product* $(xy + yx)/2$ belongs to J; J is said to be *unital* if there exists a $u \in J$ such that $ux = xu = x$ for every $x \in J$. If J is a unital Jordan algebra, a linear mapping $\Phi : J \to J$ is called a *homomorphism of unital Jordan algebras* if it preserves the Jordan product i.e.,

$$\Phi(xy + yx) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x)$$

for every $x, y \in J$.

The argument of Theorem B can be easily adapted to the Jordan algebra setting, by considering the hermitian product

$$<x, y> = \varphi\left(\frac{xy + yx}{2}\right).$$

We are then led to the following extension of Poincaré recurrence theorem for Jordan dynamical systems:

Let J be a unital Jordan algebra and let $\Phi : J \to J$ be a homomorphism of unital Jordan algebras. We assume also the existence of a positive state $\varphi \in J^\star$ such that $\varphi \circ \Phi = \varphi$.

Then

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \varphi \left(\frac{a\Phi^k(a) + \Phi^k(a)a}{2} \right) \right| > 0$$

for every $a \in J$ with $\varphi(a) \neq 0$.

3. The case of irrational rotation

The canonical model acts on the unit circle $\mathbb{T}$ (which we think of as $\mathbb{R}/\mathbb{Z}$) via the unimodular function taking t to $z(t) = e^{2\pi i t}$. Fix an irrational number θ and consider two unitaries operators on $H = L^2(\mathbb{T}) : U$, of multiplication by $z(t)$,

$$Ux(t) = z(t) \cdot x(t)$$

and V, of rotation by θ,

$$Vx(t) = x(t - \theta).$$

A simple computation shows that

$$UV = e^{2\pi i \theta} VU.$$

It turns out that $\mathcal{A}_\theta$, the $C^\star$-algebra generated by U and V, is simple and admits a unique trace τ. Notice that τ is also faithful.

We are thus led to consider the $C^\star$-dynamical system $(\mathcal{A}_\theta, \Phi, \tau)$, where

$$\Phi(A) = V^\star A V.$$

The action of Φ on the $C^\star$-algebra generated by U is that of irrational rotation, while the action on the $C^\star$-algebra generated by V is the identity. Nevertheless, the action on the whole $\mathcal{A}_\theta$ is far from obvious. Think for example that $\mathcal{A}_\theta$ has plenty of projections. A certain type of recurrence is of course expected and that is what Theorem A yields:

$$Re\,\tau\,(V^\star A^n V A) \geq |\tau(A)|^2 - \varepsilon$$

for n in a relatively dense subset of $\mathbb{N}$, which depends on A and ε.

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